Center for Algebraic Thinking

# MODULE

## *Modeling: Translating Words to Equations-The Role of Letters in Equations*

**BACKGROUND**
This module is focused on the teaching of algebra translation as it appears between words and equations. Almost all word problems begin with a problem statement in written or spoken language that requires a semantic understanding of the problem situation and task. In algebra, there is usually a translation of that understanding into the "language" and syntax of algebraic expressions and equations. There is a related, although cognitively different, skill that is required to understand the semantics of a quantitative model that is derived from the interpretation of an algebraic expression or equation. These are the areas of interest for this module, and as they relate to model building in students learning algebra.

What's the important math?

The power of algebra is that it may be used to model quantitative relationships represented in equations that represent a very clever tool for answering quantitative questions that may shortcut far lengthier guess and check routines using arithmetic. The utility of algebra depends on the validity of the algebraic equation models and the reliability of their interpretations based upon a) an understanding of the meaning of the symbols used, and b) the correct application of the rules and procedures used in the "reduction and comparison" of algebraic expressions in equations. This module considers research on the learning and teaching of the former consideration, namely the meaning of the symbols used and the syntax of algebra as related but distinct from our natural language, particularly vernacular English.

The modules on ***Translating Words to Equations*** treat five fundamentally important ideas in the teaching and learning of algebra translation:

1. use of letters to signify *quantities* that may be:
   a) letter as unknown as in $5n = 2(n-3) + 8$
   b) letter as variable as in $5n = y/8$
2) numbers as *constants* as in 'a' above (the number 8)
3) numbers as *factors* that imply operations (as in the 5 in 5n)
4) use of the *equal sign* as a statement of precise numerical equivalence only.
5) variables of interest --- the need for determination of which variables matter and the avoidance of irrelevant and extraneous variables and maintaining their definitions after algebraic transformations of equations

Finally, we consider instructional suggestions intended to overcome extremely resilient misconceptions and offer problems that may promote class discussion for successful learning of algebra translation tasks.

**1) SET: Engage with a problem or problems that help teachers consider students' algebraic thinking (teachers' prior knowledge)**
Algebra Translation Tasks for Instruction: [Note: These may be read aloud, with students working in pairs on them and completing them one at a time to be followed with class discussion. These should be completed before viewing the video of algebra student completing the same problems]:

"**Jim's marbles plus five is equal to the same number of marbles as Lucy has."**

[That is an odd statement:  How about:]

**"If Jim had five more marbles than he has, then he would have as many marbles as Lucy has."**

**"Leslie weighs 10 pounds more than Janet."**

**"Leslie weighs 10 pounds more than Janet, and Leslie weighs 137 pounds."**

**"I am just as rich as my neighbor."**

**"I am just as eloquent as my neighbor."**

**"I am much more handsome than my neighbor is."**

**"I am three times more handsome than my neighbor is."**

**"I am 24 years younger than my father."**

**"My father is twice as old as I am."**

**2) STUDENTS: Watch video clips of students describing their thinking as they engage with the student and teacher problem.  Use "Boy 3 Students Teachers" video or "Girl 5 S and T."**

What do you learn from what you are hearing or seeing regarding students' thinking?

**Interview Questions for Algebra Translation**

1. How did you think about this problem?
2. Explain why you chose each variable?
3. How do you know your equation will solve the problem?

**Algebra Translation Problems:** [This problem sequence could be used with any of the three modules to follow.  Note, these problems should be written one to a large index card and handed to the research subject one at a time to be read aloud by the subject.  The students should be asked to write whatever equations, pictures, or expressions, tables, graphs, etc. with a permanent ink sharpie on a large flipchart.  As pages get used up, they should be numbered, ripped from the pad and taped to a wall for students to refer to as needed.  Newer entries should be written large and below the earlier ones in a clear succession.  Students should not be permitted to erase or obliterate their work, but instead, they should place an 'X' next to any work that they think is mistaken or that they would like to delete.  These should be completed quickly, and the student does not need to "solve" the problem.]

The purpose of these questions is to elicit the range of thinking students may use in their initial analysis of a word problem to translate to algebra.  Any one of the problems could be probed for more information on what the student is thinking as s/he makes choices for representation.

**If you think it is possible, please use algebra to describe the following relationship:**

> "Jim's marbles plus five is equal to the same number of marbles as Lucy has."

**If you think it is possible, please use algebra to describe the following relationship:**

> "If Jim had five more marbles than he has, then he would have as many marbles as Lucy has."

**If you think it is possible, please use algebra to describe the following relationship:**

> "Leslie weighs 10 pounds more than Janet."

**If you think it is possible, please use algebra to describe the following relationship:**

> "Leslie weighs 10 pounds more than Janet, and Leslie weighs 137 pounds."

**If you think it is possible, please use algebra to describe the following relationship:**

> "I am just as rich as my neighbor."

**If you think it is possible, please use algebra to describe the following relationship:**

> "I am just as eloquent as my neighbor."

**If you think it is possible, please use algebra to describe the following relationship:**

> "I am three times more handsome than my neighbor is."

**If you think it is possible, please use algebra to describe the following relationship:**

> "I am 24 years younger than my father."

**If you think it is possible, please use algebra to describe the following relationship:**

> "My father is twice as old as I am."

**Students and Teachers Problem**

Using the letters ‘S’ for the number of students and ‘T’ for the number of teachers, write an equation to represent the following relationship: “At this school, there are twenty times as many students as there are teachers”.

**Chocolate Milk and White Milk Problem**

Using the letters ‘C’ for the number of people who drink chocolate milk and ‘W’ for the number of people who drink white milk, write an equation to represent the following relationship: “In the school cafeteria, for every four people who drink chocolate milk five people drink white milk.”

**Feet & Yards Problem:**

Write an equation that describes the relationship between the number of feet and the number of yards in any measure of length. Use “F” for the number of feet and “Y” for the number of yards. [Imagine you go out to a football field and measure some length. That length could be stated as a number of feet or a number of yards. What is the equation that describes the relationship between those two quantities.]

3) **<u>RESEARCH</u>: Examine/discuss research (encyclopedia entries)**

**Students’ Difficulties with Algebra Translation Tasks:**
There is extensive research literature on student difficulties with these tasks (see bibliography) that indicate deep and resilient misconceptions around all of the above ideas, mostly with the meaning of letters, operators, and equal signs in algebra equation writing. The research shows that many students grasp the semantic meaning conveyed in algebra translation word problems, yet make similar and important mistakes that yield impossible equations and quantitative solutions that run counter to their understandings. Often their understanding is so resilient that to create a new understanding that is closer to that of a mathematics teacher requires a deliberate attempt to confront the student with at least two cognitively compelling yet mutually contradictory solutions.

<u>The Role of Letters in Algebra Equations</u>

***Letter as Quantity:***

Among the most critically important ideas in algebra is the use of alphabet letters to represent quantities. However, letters are often used to represent many other entities in society and in mathematics. For example, we use letters to represent points, vertices, lines, angles, functions, operators, units, dimensions, etc. In equation-writing, letters must be quantities that vary or not, have a very distinct role in algebra. Often students use letters in equations to stand for other entities that may be specific or generalized. Here are some examples:

“Write an equation to represent the following relationship: I am twice as good looking as my neighbor is.”

What students may write is: I = 2 x N

Which begs the question: “Does that mean that two ugly neighbors are as good-looking as I am?”

Of course the statement above is different than the following problem: "Write an equation that represents the following relationship: I have twice the number of dandelions on my front lawn as my neighbor does." Here the use of letters to stand for quantities of dandelions is acceptable.

Students often don't make the above distinction, and use the letter to represent a broadly defined entity that only makes sense in the context stated and not in the algebraic translation. Students who correct this by stating some scale of appreciation of good looks, such as on a "scale of 1 to 10" your looks are such and such, may think they overcame the difficulty. However, the problem of writing equations with such scales is still not completely resolved. The equation may be used to solve for a quantity such as giving one value, my neighbor's score for example, and then solving for the unknown quantity denoted by the letter "I" or my score on the good-looks scale. But posing the additional question above about two ugly neighbors somehow being the same as one handsome one often tests student understanding of the meaning of the letters in a way that challenges assumptions and focuses discussion around the topic of the meaning of letters in algebra.

***Letter as Label:***

As mentioned above, another very confusing use of letters in mathematics is to use letters to stand for objects. In this sense a letter is a label as in the problem statement:

> "Write an equation to show the relationship between the number of centimeters and the number of meters in any measure of length."

Students will write: $100C = M$ and say, "there are 100 centimeters in one meter".

While this is true verbalization of fact, the above equation would generate lengths that would be quite incorrect from those measured. In fact, one would multiply the number of meters by 100 to find the number of centimeters. In the solution above, the letter C is a label that stands for "a centimeter" as the letter M is a label that stands for "one meter". As these labels are fixed definitions and designators, they may not change --- there are no "variables" in the intended equation. In this equation, any change must be effected by additional operations that the problem solver must use to arrive at the correct answer. One method for doing this will be described later, in the section about the resilience of misconceptions.

***Letter as Variable:***

The use of letters to stand for units as we see above is not the only use of letters being used as labels instead of variables. Another example of "letter as label" is when students associate a letter with a noun. For example, the well-researched Students and Professors problem illustrates several ideas that students have about variables, coefficients, and equality that are different from the way that mathematics teachers think of these:

> "Using the letters 'S' for the number of students and 'P' for the number of professors, write an equation to represent the following relationship: 'At this university, there are six times as many students as there are professors'"

Students will often write $6S = P$, and state, "For every 6 students there is 1 professor". While this statement is true and an accurate interpretation of the relationship, the apparent word-order match that the student has treats the letters S and P as labels for a student and a professor rather than as variables that may change. This occurs even though the problem statement clearly defines the letters as numbers.

Another explanation for the equation $6S = P$ that conveys the semantic meaning accurately, but uses the syntax of algebra is when students draw a column of six 'S's alongside one 'P'. They then add more groups of Ss and

Ps in the same ratio as their respective numbers increase. Their verbal explanation for the equation is reasonable to them, and they feel that by using this ratio device, they can then use their equation to find the number of students given any number of professors; and indeed, they can.

Of course, there are students who will write the correct equation: 6P = S. Their solution to the problem, explanation and verification involves treating the letters P and S as they were intended and described in the problem statement; as letters that stand for numbers whose values vary in a fixed relation to one another.

One important note: some readers may believe that the problem of the use of letters may be obviated by the use of letters that do not appear in the terms of the problem statement, for example, instead of using the letters S and P, we should use 'x' and 'y'. While some students may benefit from this, studies indicate that there are not significant differences in student response. Furthermore, given the context of most word problems and applications of algebra in real-world modeling, the use of letters related to the quantities symbolize is ubiquitous.

4) **<u>ASSESSMENT</u>: Consider assessments (Formative Assessment Database)**
[note, these may be read aloud, with students working in pairs on them and completing them one at a time to be followed with class discussion]:

**"At this school, there are 20 times more students than there are teachers."**

**"In the school cafeteria, for every four people who drink chocolate milk five people drink white milk."**

**"Write an equation that relates the number of feet to the number of yards in any measure of length."**

5) **<u>SUGGESTIONS FOR TEACHING</u>: Consider strategies based on research (including apps)**

1) Encourage students to write the definition of variables for the problem by beginning with the phrase, "let n = the number of …"
2) To avoid confusing the variable with a label, use letters that have no connection to the words in the problem.
3) Make a table of data.
4) Draw a picture or diagram of the problem situation.
5) Make a graph of the functional relationship and relate the graph to the equation.
6) Write a thought-process as the equation is created
7) Check the final equation with a pair of numbers that are known to satisfy the given problem statement.
8) Write a computer program that inputs one value and outputs the other.
9) Engage in a discussion where mental arithmetic is used to calculate the output of an input value.

While many of these ideas may prove helpful to some students, there are no guarantees that they will be successful for all students. For example, in '1' above, the 'Students and Professors' problem states specifically that the letter stands for "the number of…", yet in repeated studies, approximately a quarter of university engineering students made the reversal error. When the letters 'x' and 'y' were used, students still responded with the reversed equation. Likewise, many students would themselves derive tables of data to *support their reversed equation*, as others did with diagrams of students and professors. When the relationship stated in words was replaced with an illustration featuring a supposed aerial view of an airfield with a ratio collection of wide body planes and narrow body planes, the same reversal error occurred in near equal frequency as the word problem.

The one area that seems to eliminate the reversal error is to have students write computer programs that require explicit designation of variables as inputs and outputs and where the operations must be stated in the program (the P must be multiplied by the 6 to output S). However, the success observed with the computer program did not always generalize to the algebra. In fact, some students would write a correct computer program and an incorrect algebra equation on the same sheet of paper and *not see* the contradiction in front of them.

Finally, the idea of getting students to use their equations to input values and output corresponding values as a check is also problematic. Here is a recreation of a discussion that the author had with a student who had written the reversed equation for the following problem:

> **Feet & Yards Problem:**
>
> Write an equation that describes the relationship between the number of feet and the number of yards in any measure of length. Use "F" for the number of feet and "Y" for the number of yards. [Imagine you go out to a football field and measure some length. That length could be stated as a number of feet or a number of yards. What is the equation that describes the relationship between those two quantities.]

The student wrote, 3f = y, and said, "There are three feet in every yard." When asked if the equation could be used to find the number of feet in 30 yards, the student replied:
"Sure, this equation says there are three feet in one yard. In algebra, you don't have to put the 1 but you could. [She wrote, 3f = 1y] So this says there are three feet in one yard, but you don't have one yard, you want ten yards, so you would have to multiply the 1y by ten [student writes 3f = (10)1y], but in algebra what you do to one side of the equation, you have to do to the other side too, so I am going to multiply the 3f by ten too [student writes (10)3f = (10)1y]. When I multiply everything, you get 30f = 10y, or there are thirty feet in ten yards."

This example demonstrates the power of a conceptual framework for the use of letters, numbers, and equal signs as well as the rules of algebra equation solving, that all go toward supporting a semantic understanding of the problem and a syntactically mistaken use of algebra symbols. That the solution is consistent with other ideas in algebra *and* leads to correct answers for this type of problem *all of the time,* only serves to reinforce the misconceptions discussed here and makes teaching all the more challenging.

6) **Did the preservice teachers understand? How do you know? Evidence**

## REFERENCES


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See Algebraic Thinking Encyclopedia at http://algebraicthinking.org/algebra-thinking-references#Modeling for additional references.